On the connection between quantisation schemes and coherent states

1989 J. Phys. A: Math. Gen. 22 2751

(http://iopscience.iop.org/0305-4470/22/14/023)

J. Phys. A: Math. Gen. 22 (1989) 2751-2757. Printed in the UK

On the connection between quantisation schemes and coherent states

L C Papaloucas† and J Rembieliński‡§

† Institute of Mathematics, University of Athens, 106 79 Athens, Greece
‡ Institute of Physics, University of Lódź, Nowotki 149/153, 90-236 Lódź, Poland

Received 28 October 1988

Abstract. We analyse a possible connection between different quantisation schemes and the Bargman-Segal realisation of the Heisenberg algebra H. We show that only a one-parameter subfamily of the family of Heisenberg algebras H_Q subduced from $H \oplus H$ can be rewritten in the Bargman-Segal form.

1. Introduction

One of the most interesting approaches to the quantisation problem is that which formulates itself on the phase space of the physical system under consideration. Such a formulation is based on the observation made by several authors [1-7] who have suggested a quantisation mapping adhering to the form

$$\hat{f} = \mathrm{i}X_f + f - p\partial f/\partial p - q\partial f/\partial q \tag{1.1}$$

where X_f is a tangent vector field on phase space associated with the function f for some vector field X [8, 9].

Specific realisations of quantisation mapping resulting from the general prescription given by (1.1) are the following.

(i) Van Hove's [1, 7] mapping

$$\hat{Q} = q + \mathrm{i}\frac{\partial}{\partial p} \tag{1.2a}$$

$$\hat{P} = -\mathrm{i}\frac{\partial}{\partial q} \tag{1.2b}$$

or its improved form [10]

$$\hat{Q} = \tfrac{1}{2}q + \mathrm{i}\frac{\partial}{\partial p} \tag{1.3a}$$

$$\hat{P} = -2\mathrm{i}\frac{\partial}{\partial q}. \tag{1.3b}$$

§ Supported by CPBP 01.09.

0305-4470/89/142751+07$02.50

(ii) The symmetric quantisation mapping [4-6, 10]

$$\hat{Q} = \mathrm{i}\frac{\partial}{\partial p} + \tfrac{1}{2}q \tag{1.4a}$$

$$\hat{P} = -\mathrm{i}\frac{\partial}{\partial q} + \tfrac{1}{2}p. \tag{1.4b}$$

There are various reasons why the above prescriptions might be advantageous with respect to the conventional quantisation mapping (see [1-10] and also Chernoff [11]).

The aim of the present paper is to find a specific relation between possible quantisation schemes and the Bargmann-Segal [12, 13] realisation (hereafter referred to as BS) of the Heisenberg algebra. As is well known the last is closely connected to the coherent states [14, 15].

Our paper is organised as follows. In § 2 we investigate the Heisenberg subalgebras H_Q and H_q rotated by a symplectic transformation. We relate our findings to the irreducible subspaces connected with the eigenvalues of N_q. In § 3 we connect the realisation of H_Q with the BS one.

2. Embeddings of H in H⊕H and the representation problem

Let $\mathscr{H}$ denote the Hilbert space of square-integrable functions $\Psi(q, p)$ of q and p with measure $\mathrm{d}\mu_0 = \mathrm{d}q\mathrm{d}p$ which are defined on the phase space $\mathbb{R}^2$. Following the famous von Neumann theorem [16] the space $\mathscr{H}$ is the underlying space for irreducible representation of the Heisenberg algebra H generated by the multiplication operations by q and p by differentiation $-\mathrm{i}\partial/\partial q$ and $-\mathrm{i}\partial/\partial p$ and by identity. In H we can distinguish two subalgebras generated by $(q, -\mathrm{i}\partial/\partial q, I)$ and $(p, -\mathrm{i}\partial/\partial p, I)$, respectively, with a common one-dimensional subspace spanned by the identity I. It is obvious that the above decomposition of the algebra H is not unique: we can go to another set of generators by a symplectic transformation $\Omega \in \mathrm{Sp}(2; R)$

$$\begin{pmatrix} \hat{Q} \\ \hat{P} \\ \hat{q} \\ \hat{p} \end{pmatrix} = \Omega \begin{pmatrix} q \\ -\mathrm{i}\partial/\partial q \\ p \\ -\mathrm{i}\partial/\partial p \end{pmatrix} \tag{2.1}$$

where

$$\Omega^{\mathrm{T}} J \Omega = J \tag{2.2a}$$

$$\Omega^* = \Omega \tag{2.2b}$$

and

$$J = \left(\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right) \tag{2.3}$$

is the symplectic matrix.

The pair $(\hat{Q}, \hat{P}, I)$ and $(\hat{q}, \hat{p}, I)$ now generate two Heisenberg algebras H_Q and H_q, respectively. Each symplectic transformation $\Omega \in \mathrm{Sp}(2; R)$ can be decomposed into two parts; one part which leaves invariant the subalgebras spaces (it changes the basis within subalgebras only) and the other part which non-trivially mixes these subspaces. The former one forms the stability group of a given subalgebra and is given by the evident condition

$$(I - \Pi)\Omega_0\Pi = 0 \tag{2.4}$$

where $\Omega_0 \in G_0$, the stability subgroup ($G_0 \subset \mathrm{Sp}\,(2; R)$), I is the identity matrix,

$$\Pi = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.5}$$

projects on the upper subspace corresponding to H_Q and I_2 is the 2×2 unit matrix. The condition (2.4) can be solved immediately and we obtain as a result

$$\Omega_0 = \begin{pmatrix} \Omega_+ & 0 \\ 0 & \Omega_- \end{pmatrix} \tag{2.6}$$

where $\Omega_\pm$ are 2×2 real matrices with $\det \Omega_\pm = 1$, i.e.

$$\Omega_0 \in \mathrm{Sp}(1; R)\times \mathrm{Sp}(1; R) \simeq \mathrm{SL}(2; R)\times \mathrm{SL}(2; R) = G_0. \tag{2.7}$$

Now the essentially different choices of the subalgebra, say H_Q, are parametrised by points of the quotient space [17, 18] $\mathrm{Sp}(2; R)/\mathrm{Sp}(1; R)\times \mathrm{Sp}(1; R)$. In order to obtain an explicit parametrisation of the above coset space, let us note that the elements outside the Lie algebra of the stability subgroup take the form

$$\begin{pmatrix} 0 & -\sigma_2\omega^{\mathrm{T}}\sigma_2 \\ \omega & 0 \end{pmatrix} \tag{2.8}$$

where

$$\sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \tag{2.8a}$$

is the Pauli matrix and ω is an arbitrary 2×2 real matrix. Consequently the corresponding group elements are

$$W = \left(\begin{array}{cc|cc} \alpha & 0 & \beta & \gamma \\ 0 & \alpha & \delta & -\varepsilon \\ \hline \varepsilon & \gamma & \alpha & 0 \\ \delta & -\beta & 0 & \alpha \end{array}\right) \tag{2.9}$$

where the real parameters α, β, γ, δ, ε satisfy the relation

$$\alpha^2 - (\beta\varepsilon + \gamma\delta) = 1. \tag{2.10}$$

Note that the elements W and $-W$ belong to the same coset because $\pm I \in G_0$. Therefore, to get a global homeomorphy between the coset space $\mathrm{Sp}(2; R)/\mathrm{Sp}(1; R)\times \mathrm{Sp}(1; R)$ and the set $\{W\}$ we must demand $\alpha \geqslant 0$. Geometrically the considered quotient space is the one-sheet hyperboloid $H_{3,2}$ (dim $H_{3,2}=4$); this yields readily if we pass to coordinates $x_1, \ldots, x_5$ defined as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \varepsilon \end{pmatrix}. \tag{2.11}$$

In terms of these coordinates, the constraint (2.10) takes the form

$$x_1^2+x_2^2+x_3^2-x_4^2-x_5^2=1 \tag{2.12}$$

determining $H_{3,2}$.

We are led to the conclusion that $\Omega \in \mathrm{Sp}(2; R)$ can be represented as follows:

$$\Omega = \Omega_0 W \tag{2.13}$$

with Ω_0 and W given by (2.6) and (2.9), respectively. Therefore, the explicit form of (2.1) is

$$\begin{pmatrix} \hat{Q} \\ \hat{P} \end{pmatrix} = \Omega_+ \left[\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} q \\ -\mathrm{i}\partial/\partial q \end{pmatrix} + \begin{pmatrix} \beta & \gamma \\ \delta & -\varepsilon \end{pmatrix} \begin{pmatrix} p \\ -\mathrm{i}\partial/\partial p \end{pmatrix} \right] \tag{2.14a}$$

$$\begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix} = \Omega_- \left[\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} p \\ -\mathrm{i}\partial/\partial p \end{pmatrix} + \begin{pmatrix} \varepsilon & \gamma \\ \delta & -\beta \end{pmatrix} \begin{pmatrix} q \\ -\mathrm{i}\partial/\partial q \end{pmatrix} \right]. \tag{2.14b}$$

Because we are interested in quantisation of a one-dimensional system we must subduce the representation of H acting on $\mathscr{H}$ to, say, H_Q. However the subduced representation of H_Q is in $\mathscr{H}$ highly reducible; irreducible subspaces are marked by eigenvalues of the occupation number operator for H_q, namely by

$$N_q = \tfrac{1}{2}(\hat{q}-\mathrm{i}\hat{p})(\hat{q}+\mathrm{i}\hat{p}). \tag{2.15}$$

In the following we restrict ourselves to the irreducible subspace connected with the eigenvalue zero of N_q. To do this, we note first that the equation

$$N_q \Psi(q, p) = 0 \tag{2.16}$$

implies, via positive definiteness of the norm in $\mathscr{H}$, that

$$(\hat{q}+\mathrm{i}\hat{p})\Psi(q, p) = 0. \tag{2.17}$$

Therefore the explicit form of the projection equation (2.17), using (2.14a, b), is

$$\begin{aligned} 0 = \{\alpha[(a+\mathrm{i}c)p+(d-\mathrm{i}b)\partial/\partial p]+[\varepsilon(a+\mathrm{i}c)+\delta(b+\mathrm{i}d)]q \\ -\mathrm{i}[\gamma(a+\mathrm{i}c)-\beta(b+\mathrm{i}d)]\partial/\partial q\}\Psi(q, p) \end{aligned} \tag{2.17a}$$

where a, b, c, d parametrise Ω_-:

$$\Omega_- = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.18}$$

with the constraint

$$ad - bc = 1. \tag{2.19}$$

The general case will be considered in the next section.

3. Connection with the BS representation

Now let us try to connect the realisation of $H_Q(\alpha, \beta, \gamma, \delta, \varepsilon)$ with the BS one. We remember first that in the BS representation case we have to deal with the Hilbert space of entire functions $\phi(z^*)$ of z^*, $z \in \mathbb{C}$, with the measure

$$d\mu = d^2z \exp(-\tfrac{1}{4}|z|^2) \tag{3.1}$$

and with action of the annihilation and creation operators corresponding to the differentiation $\partial/\partial z^*$ and multiplication by z^* respectively. Therefore we should redefine Ψ and $d\mu_0$ as well as connect z with q and p as follows:

$$\Psi(q, p) = \exp(-\tfrac{1}{2}|z|^2)\phi(z^*) \tag{3.2a}$$

$$d\mu_0 = d\mu \exp(\tfrac{1}{4}|z|^2) \tag{3.2b}$$

$$z = \frac{1}{\sqrt{2}}(q' + ip') \tag{3.2c}$$

where q' and p' are appropriate linear combinations of q and p. Moreover, in the action on $\phi(z^*)$

$$\frac{1}{\sqrt{2}}(\hat{Q} + i\hat{P}) = \frac{\partial}{\partial z^*} \tag{3.2d}$$

$$\frac{1}{\sqrt{2}}(\hat{Q} - i\hat{P}) = z^*. \tag{3.2e}$$

However the consistency between (2.17) and (3.2*a*–*e*) leads to the relations

$$\alpha = \frac{1}{\sqrt{2}} \qquad \beta = 0 \qquad \delta = -\frac{1}{2\gamma} \qquad \varepsilon = 0 \tag{3.3a}$$

$$\Omega_+ = -\frac{1}{\sqrt{2}}\begin{pmatrix} d/\gamma & 2\gamma c \\ b/\gamma & 2\gamma a \end{pmatrix} \tag{3.3b}$$

$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} -d/\gamma & \sqrt{2}\, c \\ -b/\gamma & \sqrt{2}\, a \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} \tag{3.3c}$$

with

$$\Omega_- = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.4}$$

Consequently, the explicit form of the base elements of the distinguished Heisenberg algebra H_Q is

$$\hat{Q} = -\frac{d}{2\gamma} q + \frac{c}{\sqrt{2}} p + i\frac{d}{\sqrt{2}}\frac{\partial}{\partial p} + i\gamma c\frac{\partial}{\partial q} \tag{3.5a}$$

$$\hat{P} = -\frac{b}{2\gamma} q + \frac{a}{\sqrt{2}} p + i\frac{b}{\sqrt{2}}\frac{\partial}{\partial p} + i\gamma a\frac{\partial}{\partial q} \tag{3.5b}$$

while the constraint (2.17*a*) for $\Psi(q, p)$ takes the form

$$[(a + ic)(p - i\gamma\sqrt{2}\,\partial/\partial q) - (b + id)(1/\gamma\sqrt{2}q + i\partial/\partial p)]\Psi(q, p) = 0 \tag{3.6}$$

with (3.2*a*) as the solution and z given by

$$z=-\frac{1}{\gamma\sqrt{2}}(d+\mathrm{i}b)q+(c+\mathrm{i}a)p. \tag{3.7}$$

From the above derivation we see that only the one-parameter subfamily of the Heisenberg algebra family $\mathrm{H}_Q(\alpha, \beta, \gamma, \delta, \varepsilon)$, namely those corresponding to $H_Q(1/\sqrt{2}, 0, \gamma, -1/2\gamma, 0)$, can be connected with the BS realisation and consequently with the coherent-state description

$$\Psi(q, p)=\langle z|\Psi\rangle \tag{3.8}$$

where $\{|z\rangle\}$ is the set of coherent states. Note that for

$$\gamma=-\sqrt{2} \qquad a=1/\sqrt{2} \qquad b=c=0 \qquad d=\sqrt{2} \tag{3.9}$$

we obtain from (3.5*a*, *b*) the symmetric form of $\hat{Q}$ and $\hat{P}$

$$\hat{Q}_s=\tfrac{1}{2}q+\mathrm{i}\partial/\partial p \tag{3.10a}$$

$$\hat{P}_s=\tfrac{1}{2}p-\mathrm{i}\partial/\partial q. \tag{3.10b}$$

as well as

$$z=\frac{1}{\sqrt{2}}(q+\mathrm{i}p). \tag{3.11}$$

On the other hand it is very surprising that the van Hove and improved van Hove [10] choice of $\hat{P}$ and $\hat{Q}$ does not belong to the above subfamily and consequently cannot be rewritten in the BS form.

Now we investigate the general case, that is when (2.16) takes the form

$$(N_q-n)\Psi(q, p)=0 \tag{3.12}$$

with $n\neq 0$. This equation is the projection equation on a subspace of $\mathscr{H}$ underlying an irreducible representation of H_Q. If eventually this representation of the Heisenberg algebra can be rewritten in the BS form, then in the action on $\Psi(q, p)=\Psi'(z, z^*)$ the generators

$$\frac{1}{\sqrt{2}}(\hat{Q}+\mathrm{i}\hat{P}) \tag{3.13a}$$

and

$$\frac{1}{\sqrt{2}}(\hat{Q}-\mathrm{i}\hat{P}) \tag{3.13b}$$

should have the form

$$\tfrac{1}{2}z+\partial/\partial z^* \qquad \text{and} \qquad z^* \tag{3.14}$$

respectively.

On the other hand (N_q-n), from construction, is a differential operator of degree two with respect to $\partial/\partial z$ and $\partial/\partial z^*$. Moreover N_q-n should commute with $\frac{1}{2}z+\partial/\partial z^*$ and z^*.

However, a direct calculation shows that this holds only for the above-considered case $n=0$. Therefore, for $n>0$ no representation of H_Q $(\alpha, \beta, \gamma, \delta, \varepsilon)$ in $\mathscr{H}$ related to the BS realisation.

References

[1] van Hove L 1951 *Proc. R. Acad. Sci. Belg.* **26** 317
[2] Segal I E 1960 *J. Math. Phys.* **1** 468
[3] Sourian J M 1970 *Structure des Systemés Dynamiques* (Paris: Dunod)
[4] Streater R F 1966 *Commun. Math. Phys.* **2** 354
[5] Berezin F A and Subin M A 1972 *Colloq. Math. Soc. Janos Rolyai* vol 5 (Amsterdam: North-Holland)
[6] George C and Prigogine I 1979 *Physica* **99A** 369
[7] Prugovecki E 1982 *Phys. Rev. Lett.* **49** 1065
[8] Joseph A 1970 *Commun. Math. Phys.* **17** 210
[9] Wollenberg I S 1967 *Proc. Am. Math. Soc.* **20** 315
[10] Ktorides C N and Papaloucas L C 1986 *Prog. Theor. Phys.* **75** 301
[11] Chernoff P R 1981 *Hadronic J.* **4** 479
[12] Bargmann V 1961 *Commun. Pure Appl. Math.* **14** 187
[13] Segal I E 1962 *Illinois J. Math.* **6** 500
[14] Klauder R J 1960 *Ann. Phys., NY* **11** 123
[15] Glauber J R 1963 *Phys. Rev.* **131** 2766
[16] von Neumann J 1955 *Mathematical Foundations of Quantum Mechanics* (Princeton, NJ: Princeton University Press)
[17] Barut A and Raczka R 1977 *Theory of Group Representations and Applications* (Warsaw: Polish Scientific Publishers)
[18] Helgason S 1978 *Differential Geometry, Lie Groups and Symmetric Spaces* (New York: Academic)